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Vafa's formula and equivariant K-theory

Kaoru Ono [a,b] and Shi-shyr Roan [a,c]

[a] *Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, 5300 Bonn 3, Germany*
[b] *Department of Mathematics, Faculty of Science, Ochanomizu University, Otsuka, Tokyo 112, Japan*
[c] *Institute of Mathematics, Academia Sinica, Taipei, Taiwan*

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Vafa's formula of the Euler characteristic of Calabi–Yau hypersurface in weighted projective four-space is identified with the Euler characteristic of S^1-equivariant K-theory of the Seifert fibration over it.

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1. Introduction

For a finite group G acting on an n-dimensional complex manifold M, the corrected Euler characteristic for the quotient M by G in string theory is the following expression of the orbifold Euler characteristic [2]:

$$\chi(M,G) = \frac{1}{|G|} \sum_{gh=hg} \chi(M^{\langle g,h \rangle}),$$

where the summation runs over all commuting pairs (g,h) of elements g and h in G and $M^{\langle g,h\rangle}$ denotes the common fixed point set of g and h. Atiyah and Segal [1] noticed that the above expression of $\chi(M,G)$ equals the Euler characteristic of the equivariant K-theory $K_G^*(M)$. In the case when M/G has a resolution $\widehat{M/G}$ with trivial canonical bundle, $\chi(M,G)$ is expected to be the same as $\chi(\widehat{M/G})$. This statement holds for many interesting cases, e.g., dim $M = 2$ [3], or dim $M = 3$ with abelian group G [4,5]. This paper deals with a similar situation, the equivariant K-theory for the action of the circle group.

In the study of the conformal field theory of the Landau–Ginzburg model, Vafa has obtained the following expression of Witten's index [9]:

$$\mathrm{Tr}\,(-1)^F = \frac{1}{d} \sum_{l,r=0}^{d-1} \prod_{lq_i, rq_i \in \mathbb{Z}} (1 - 1/q_i),$$

where d is the degree of the superpotential $f(z_0, \ldots, z_m)$ with weight $(z_i) = n_i$, q_i $(i = 0, \ldots, m)$ is the charge n_i/d of z_i with $\sum_{j=0}^m q_j = 1$.

Vafa's formula of Witten's index has a topological interpretation on the zero locus X of the polynomial $f(z_0, \ldots, z_m)$ in the weighted projective m-space $WP^m_{(n_i)}$. For $m = 4$, the minimal toroidal resolution $\hat{X}$ of X is a Calabi–Yau space. It is shown in ref. [6] that the Euler number $\chi(\hat{X})$ is expressed by the above quantity in Vafa's formula. Atiyah suggests that the connection of Vafa's formula and the equivariant K-theory exists as the case of the action of a finite group. The main result of this paper is to show that this is indeed true.

Let $WP^m_{(n_0,\ldots,n_m)}$ be the m-dimensional weighted projective space with weights $(n_0, \ldots, n_m)$ satisfying g.c.d.$\{n_i | i \neq j\} = 1$ for all j. Denote $D = \sum_{i=0}^m n_i$. Consider the natural projection $\mathbb{C}^{m+1} - 0 \to WP^m_{(n_i)}$, and restrict it to the unit sphere S^{2m+1}. We get a Seifert fibration $S^{2m+1} \to WP^m_{(n_i)}$. For a subset A of $WP^m_{(n_i)}$, we denote by $S_A \to A$ the restriction of the Seifert fibration to A. Our main result is the following

Theorem 1.1. *Let X be a quasi-smooth hypersurface in $WP^m_{(n_i)}$ defined by a quasi-homogeneous polynomial of degree d. Then $K^0_{S^1}(S_X)$ and $K^1_{S^1}(S_X)$ are of finite rank and the following equality holds:*

$$\operatorname{rank} K^0_{S^1}(S_X) - \operatorname{rank} K^1_{S^1}(S_X) = (D-d) + \frac{1}{d} \sum_{l,r=0}^{d-1} \prod_{lq_i, rq_i \in \mathbb{Z}} (1 - 1/q_i),$$

where $q_i = n_i/d$.

2. Preliminaries

In this section, we review some facts on the equivariant K-theory. Let G be a compact Lie group and X a compact G-space. We shall denote the Euler characteristic of the equivariant K-theory of a G-space X as $\chi^K_G(X)$.

Fact 1. If G acts on X trivially, we have

$$K^*_G(X) \cong K^*(X) \otimes R(G),$$

where $R(G)$ is the representation ring of G.

Fact 2. If G acts on X freely, we have

$$K^*_G(X) \cong K^*(X/G).$$

It is well known that we can obtain the Mayer–Vietoris sequence by a routine diagram chasing argument, once we have the exact sequence of pairs and the excision property, which are found in ref. [8].

Fact 3 (Mayer–Vietoris sequence). Let X be a compact G-space and A and B are closed G-invariant subspaces such that $A \cup B = X$. Then we have the following six-term exact sequence:

$$\begin{array}{ccccc} K_G^0(X) & \to & K_G^0(A) \oplus K_G^0(B) & \to & K_G^0(A \cap B) \\ \uparrow & & & & \downarrow \\ K_G^1(A \cap B) & \leftarrow & K_G^1(A) \oplus K_G^1(B) & \leftarrow & K_G^1(X) \end{array} .$$

Consequently, $\chi_G^K(X) + \chi_G^K(A \cap B) = \chi_G^K(A) + \chi_G^K(B)$.

We also need the following

Proposition 2.1. *Let N be a finite subgroup of an abelian group G and X a G-space. Suppose the G-action on X is factored through the homomorphism $G \to G' = G/N$. Then we have*

$$K_G^*(X) \cong K_{G'}^*(X) \otimes R(N).$$

Proof. Denote by $\hat{N}$ the set of all irreducible representations of N. Since G is abelian, every irreducible representation of N can be regarded as the restriction of a certain representation of G. For each irreducible representation ρ of N, we fix an extension $\tilde{\rho}$ of ρ to the representation of G.

For a G-equivariant vector bundle E on X, $\mathrm{Hom}_N(\tilde{\rho}, E)$ denotes a G-equivariant vector bundle defined as follows:

$$\mathrm{Hom}_N(\tilde{\rho}, E) = \bigcup_{x \in X} \mathrm{Hom}_N(\tilde{\rho}, E_x).$$

$\mathrm{Hom}_N(\tilde{\rho}, E_x)$ is nothing but $\mathrm{Hom}_N(\rho, E_x)$ as N-space. However, $\mathrm{Hom}_N(\tilde{\rho}, E)$ carries a G-action in a natural way:

$$(g \cdot f)(v) = gf(g^{-1}v),$$

for $f \in \mathrm{Hom}_N(\tilde{\rho}, E)$ and v an element in the $\tilde{\rho}$-representation space. Since f commutes with the N-action, N acts trivially on $\mathrm{Hom}_N(\tilde{\rho}, E)$, i.e., $\mathrm{Hom}_N(\tilde{\rho}, E)$ is a G'-vector bundle. We define a homomorphism $\phi : K_G^0(X) \to K_{G'}^0(X) \otimes R(N)$ as follows:

$$\phi(E) = \bigoplus_{\rho \in \hat{N}} \mathrm{Hom}_N(\tilde{\rho}, E) \otimes \rho.$$

We also define a homomorphism $\psi : K_{G'}^0(X) \otimes R(N) \to K_G^0(X)$ by

$$\psi(F \otimes \rho) = F \otimes \tilde{\rho}.$$

It is clear that ϕ and ψ are inverse to each other. The case of an odd degree equivariant K-group is reduced to the case of even degree. This completes the proof. □

Remark 2.2. Note that ϕ and ψ are not ring homomorphisms.

In general, $K_G^*(X)$ is not necessarily of finite rank. However, under the condition that every isotropy group is finite, the above fact 3 and proposition 2.1 imply that it is of finite rank. Finally we recall the Chern character isomorphism for ordinary K-theory.

Fact 4. The Chern character induces an isomorphism after tensoring $\mathbb{Q}$.

$$\mathrm{ch} : K^*(X) \otimes \mathbb{Q} \to H^*(X) \otimes \mathbb{Q}.$$

3. Proof of theorem 1.1

Let X be a quasi-smooth hypersurface in $WP^m_{(n_0,\ldots,n_m)}$ defined by the zeros of a quasi-homogeneous polynomial f with degree d. We write

$$Y = \{[x_i, w] \in WP^{m+1}_{(n_i,1)} | w^d = f(x_0, \ldots, x_m)\}.$$

X is identified with the intersection of Y and $WP^m_{(n_i)}$, which is defined by the equation $w = 0$. The complement of X in Y is the Milnor fiber $F = \{(x_0, \ldots, x_m) \in \mathbb{C}^{m+1} | f(x_0, \ldots, x_m) = 1\}$. We have the diagram

$$\begin{array}{ccccc} X & \subset & Y & \supset & F \\ \downarrow & & \downarrow & & \downarrow \\ X & \subset & WP^m_{(n_i)} & \supset & U \end{array}.$$

U is the quotient of F by the monodromy map h of F,

$$h : [x_0, \ldots, x_m, w] \mapsto [x_0, \ldots, x_m, \omega^{-1} w] = [\omega^{n_0} x_0, \ldots, \omega^{n_m} x_m, w],$$

where ω is the primitive dth root of unity.

For a subgroup H of G, write

$$M^H = \bigcap \{M^g | g \in H\},$$
$$M(H) = M^H - \bigcup \{M^K | H \text{ a proper subgroup of K}\};$$

hence $M(H)$ consists of all the points of M with H as the isotropy subgroup.

Lemma 3.1. *Let G be a compact abelian Lie group and P a compact differentiable manifold with $G \times S^1$-action. Suppose the $G \times S^1$-isotropy subgroups at points of P are all finite. Then P, as a G-space, has the vanishing G-equivariant K-theory Euler characteristic, i.e., $\chi_G^K(P) = 0$.*

Proof. P, as G-space, is the union of all $P(H)$ for $H < G$. Stratify P as a finite sequence of $G \times S^1$-invariant closed subspaces

$$\phi = P_{-1} \subset P_0 \subset P_1 \subset \cdots \subset P_N = P$$

such that $P_j - P_{j-1}$ is $P(H)$ for some $H < G$. It is easy to see that there is a $G \times S^1$-invariant regular neighborhood Q_j of P_{j-1} in P_j. ($Q_0 = \phi$.) Let $P'(H) = P_j - Q_j$. By the Mayer–Vietoris sequence argument (cf. fact 3 in section 2), we have

$$\chi_G^K(P_j) = \chi_G^K(P'(H)) + \chi_G^K(P_{j-1}) - \chi_G^K(\partial Q_j).$$

By proposition 2.1 and fact 2 in section 2, $K_G^*(P'(H))$ and $K_G^*(\partial Q_j)$ are isomorphic to $K_G^*(P'(H)/G) \otimes R(H)$ and $K_G^*(\partial Q_j/G) \otimes R(H)$, respectively. Since $P'(H)/G$ and $\partial Q_j/G$ are Seifert manifolds (i.e. manifolds with non-vanishing vector fields which generate S^1-actions), they have zero Euler characteristic. By fact 4 in section 2, we have $\chi_G^K(P'(H)) = \chi_G^K(\partial Q_j) = 0$. By induction, we have $\chi_G^K(P) = 0$. □

We are going to prove the following lemmas using the above result on Seifert fibration S_A over A with $G = S^1$ which acts on fibers.

Lemma 3.2. $\chi_{S^1}^K(S_{WP^m_{(n_i)}}) = D.$

Proof. The $G(= S^1)$-action on S^{2m+1} is determined by weights $\{n_i\}$

$$(z_0, z_1, \ldots, z_m) \mapsto (\lambda^{n_0} z_0, \lambda^{n_1} z_1, \ldots, \lambda^{n_m} z_m).$$

The other S^1-action is determined by

$$(z_0, z_1, \ldots, z_m) \mapsto (t^{a_0} z_0, t^{a_1} z_1, \ldots, t^{a_m} z_m),$$

hence an action on $WP^m_{(n_i)}$:

$$[z_0, z_1, \ldots, z_m] \mapsto [t^{a_0} z_0, t^{a_1} z_1, \ldots, t^{a_m} z_m].$$

For generic integers $a_0, a_1, \ldots, a_m$, the points of P with finite $G \times S^1$-isotropy subgroup are exactly those outside fibers over $F (:= \{[1,0,\ldots,0], [0,1,\ldots,0], \ldots, [0,0,\ldots,1]\}$. Let $N(F)$ be an S^1-invariant regular neighborhood of F in $WP^m_{(n_i)}$. The Mayer–Vietoris sequence implies

$$\chi_{S^1}^K(S_{WP^m_{(n_i)}}) = \chi_{S^1}^K(S_F) + \chi_{S^1}^K(S_{WP^m_{(n_i)}} - S_{N(F)}) - \chi(S_{\partial N(F)}).$$

By proposition 2.1, the first term on the right hand side is D, and lemma 3.1 assures that the last two terms on the right hand side are zero. Hence we obtain this lemma. □

Lemma 3.3. *Let V be the complement of a regular neighborhood of X in $WP^m_{(n_i)}$. Then we have*

$$\chi_{S^1}^K(S_{WP^m_{(n_i)}}) = \chi_{S^1}^K(S_X) + \chi_{S^1}^K(S_V).$$

Proof. Since S_X is an S^1-invariant submanifold of $S_{WP^m_{(n_i)}}$, there is an S^1-invariant tubular neighborhood $N(S_X)$ of S_X. Since the real codimension of X in $WP^m_{(n_i)}$ is 2, the boundary $\partial N(S_X)$ is an S^1-equivariant circle bundle over S_X. Lemma 3.1 implies that $\chi^K_{S^1}(S_{\partial N(S_X)}) = 0$. Hence the conclusion follows from the Mayer–Vietoris sequence argument. □

It is easy to see that the group generated by the monodromy transformation h on F is of order d, and every isotropy subgroup H of the S^1-action on S_V is a subgroup of $\langle h \rangle$. Then S_V/S^1 is homotopically equivalent to the quotient space $F(H)/\langle h \rangle$. Using proposition 2.1 and lemma 3.1 we can show the following

Lemma 3.4.

$$\chi^K_{S^1}(S_V) = \sum_{H < \langle h \rangle} |H| \cdot \chi(F(H)/\langle h \rangle).$$

Proof. We stratify S_V as follows:

$$Y_0 \subset Y_1 \subset \cdots \subset Y_N = S_V$$

such that

(i) Y_j is an S^1-invariant closed subset;
(ii) $Y_j - Y_{j-1} = S_V(H_j)$ for some $H_j < \langle h \rangle$;
(iii) $\overline{Y_j - Y_{j-1}} \cap Y_{j-1}$ is a union of $S_V(H')$ for $H' \gneq H_j$.

It is easy to see that there is an S^1-invariant regular neighborhood N_j of Y_{j-1} in Y_j. Since $\partial N_j/S^1$ is a compact odd dimensional manifold, its Euler number is 0. Fact 3 in section 2 yields

$$\begin{aligned}\chi^K_{S^1}(Y_j) &= \chi^K_{S^1}(N_j) + \chi^K_{S^1}(S_V(H_j)) \\ &= \chi^K_{S^1}(Y_{j-1}) + \chi^K_{S^1}(S_V(H_j));\end{aligned}$$

hence

$$\begin{aligned}\chi^K_{S^1}(S_V) &= \sum_{H<\langle h \rangle} \chi^K_{S^1}(S_V(H)) \\ &= \sum_{H<\langle h \rangle} |H| \chi(S_V(H)/S^1) \\ &= \sum_{H<\langle h \rangle} |H| \chi(F(H)/\langle h \rangle).\end{aligned}$$

□

Lemma 3.5. *For $l = 0, 1, \ldots, d-1$, let F^{h^l} be the fixed point manifold for h^l. Then*

$$1 - \chi(F^{h^l}/\langle h \rangle) = \frac{1}{d} \sum_{r=0}^{d-1} \prod_{lq_i, rq_i \in \mathbb{Z}} (1 - 1/q_i).$$

Proof. The same as lemma 3 of ref. [6]. □

Proof of theorem 1.1. For a subset I of $\{0, 1, \ldots, m\}$, we have

$$f(z|z_i = 0, i \in I) \text{ is a non-trivial polynomial in } z_j \ (j \notin I)$$
$$\iff F_I' (:= F \cap \{[x_0, \ldots, x_m, 1] | x_i = 0 \text{ for } i \in I,\ x_j \neq 0 \text{ for } j \notin I\}) \neq \phi .$$

Write $\mathcal{I} = \{I | F_I' \neq \phi\}$, $H(I)$ is the isotropy subgroup of $\langle h \rangle$ for points in F_I', $I \in \mathcal{I}$. Then the order of $H(I)$ is equal to $c_I (:= \text{g.c.d.}\{n_j | j \notin I\})$. For a subgroup H of $\langle h \rangle$, we have

$$F(H) = \bigcup \{F_I' | H(I) = H,\ I \in \mathcal{I}\};$$

hence

$$F(H)/\langle h \rangle = \bigcup \{F_I'/\langle h \rangle | H(I) = H,\ I \in \mathcal{I}\}.$$

Let $U_I = \{[x_0, \ldots, x_m] \in WP^m_{(n_i)} - X | x_i = 0 \text{ for } i \in I\}$, $U_I' = U_I - \bigcup_{I \subsetneq J} U_J$. Then $F_I'/\langle h \rangle = U_I'$, $F(H)/\langle h \rangle = \bigcup \{U_I' | H(I) = H,\ I \in \mathcal{I}\}$. By lemmas 3.2, 3.3, 3.4, and 3.5,

$$\begin{aligned}
\chi^K_{S^1}(S_X) &= D - \sum_{H < \langle h \rangle} |H| \chi(F(H)/\langle h \rangle) \\
&= D - \sum_{I \in \mathcal{I}} \chi(U_I') c_I \\
&= D - d + d - \sum_{l=0}^{d-1} \left(\sum \{\chi(U_J') | F^{h^l}/\langle h \rangle \supseteq U_J\} \right) \\
&= D - d + d - \sum_{l=0}^{d-1} \chi(F^{h^l}/\langle h \rangle) \\
&= D - d + \sum_{l=0}^{d-1} (1 - \chi(F^{h^l}/\langle h \rangle)) \\
&= D - d + \frac{1}{d} \sum_{l,r=0}^{d-1} \prod_{lq_i, rq_i \in \mathbb{Z}} (1 - 1/q_i). \qquad \Box
\end{aligned}$$

Remark 3.6. First we compare the approach of ref. [1] and ours. The main tool in ref. [1] is the following

Fact 5. Let G be a finite group and X a compact G-space. Then there is an isomorphism:

$$K^*_G(X) \otimes \mathbb{C} \cong \bigoplus_{[g]} [K(X^g) \otimes \mathbb{C}]^{Z_g},$$

where $[g]$ is the conjugate class in G and Z_g is the centralizer of g.

To interpret the right hand side, we introduce the following space.

$$\mathcal{X} := \{(x,h) \in X \times G \mid h \cdot x = x\}.$$

G acts on $\mathcal{X}$ naturally as follows:

$$g \cdot (x,h) := (g \cdot x, ghg^{-1}).$$

Then it is easy to see that

$$\mathcal{X}/G = \coprod_{[g]} X^g/Z^g.$$

Therefore the above fact implies that the equivariant Euler characteristic of X equals the Euler characteristic of $\mathcal{X}/G$. On the other hand, $\mathcal{X}$ is also decomposed into subspaces according to the isotropy types. Our approach can be seen as the latter one.

Remark 3.7. The equivariant K-theory interpretation of Vafa's formula we have given in the main theorem is based on the action of the abelian group generated by S^1 and monodromy group $\langle h \rangle$. The same argument works also for the cases of any finite abelian group commuting with the S^1-action, instead of $\langle h \rangle$. Hence we can also obtain a similar K-theory interpretation of the generalized Vafa formula related to Calabi–Yau mirror manifolds treated in ref. [7].

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References

[1] M. Atiyah and G. Segal, On the equivariant Euler characteristics, J. Geom. Phys. 6 (1989) 671–677.

[2] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds, Nucl. Phys. B 261 (1985) 678; Strings on orbifolds II, Nucl. Phys. B 274 (1986) 285.

[3] F. Hirzebruch and T. Höfer, On the Euler number of an orbifold, Math. Ann. 286 (1990) 255–260.

[4] D.G. Markushevich, M.A. Olshanetsky and A.M. Perelomov, Description of a class of superstring compactifications related to semi-simple Lie algebras, Commun. Math. Phys. 111 (1987) 247–274.

[5] S.S. Roan, On the generalization of Kummer surfaces, J. Diff. Geom. 30 (1989) 523–537.

[6] S.S. Roan, On Calabi–Yau orbifolds in weighted projective spaces, Intern. J. Math. 1 (1990) 211–232.

[7] S.S. Roan, The mirror of Calabi–Yau orbifold, Intern. J. Math. 2 (1991) 439–455.

[8] G. Segal, Equivariant K-theory, Publ. Math. IHES 34 (1968) 129–151.

[9] C. Vafa, String vacua and orbifoldized LG models, Mod. Phys. Lett. A 4 (1989) 1169.